Correcting Finite Element Models Using a Symmetric Eigenstructure Assignment Technique

D. C. Zimmerman*
University of Florida, Gainesville, Florida
and
M. Widengren†
Royal Institute of Technology, Stockholm, Sweden

Improvement of structural models by incorporating measured structural modal parameters is approached from a controls aspect. The approach is developed for linear structures that exhibit nonproportional damping. Residual damping and stiffness matrices are determined such that the improved analytical model eigenstructure matches more closely that obtained experimentally. The method is based on the development of a symmetric eigenstructure assignment algorithm. Examples will be presented that demonstrate the algorithm.

Nomenclature

A	= state matrix, $2n \times 2n$
$\tilde{A}$	= transform of A, $2n \times 2n$
A_i	= terms of the generalized Riccati equation, $m \times m$
A_ℓ	= partition of $\tilde{A}$, $(2n - m) \times 2n$
A_u	= partition of $\tilde{A}$, $m \times 2n$
$\boldsymbol{a}_i$	= ith eigenvector of E, $2m \times 1$
B	= state control influence matrix, $2n \times m$
$\tilde{B}$	= transform of B, $2n \times m$
B_0	= control influence matrix, $n \times m$
$\boldsymbol{b}_i$	= upper partition of $\boldsymbol{a}_i$, $m \times 1$
C_0, C_1	= output influence matrices, $r \times n$
$\boldsymbol{c}_i$	= lower partition of $\boldsymbol{a}_i$, $m \times 1$
D	= damping matrix, $n \times n$
D_a	= adjusted damping matrix, $n \times n$
D_i	= transformed subspace, $(n - s) \times 1$
$\boldsymbol{d}_i$	= unmeasured components of $\boldsymbol{v}_i$, $(n - s) \times 1$
E	= Euler matrix, $2m \times 2m$
F	= feedback gain matrix, $m \times r$
G_i	= matrix, $m \times m$
I_m	= identity matrix, $m \times m$
J	= cost function
K	= stiffness matrix, $n \times n$
K_a	= adjusted stiffness matrix, $n \times n$
L_i	= subspace of ith achievable eigenvector, $n \times m$
$\tilde{L}_i$	= transformed subspace (measured), $s \times m$
M	= mass matrix, $n \times n$
m	= number of control actuators, $(m = r = 2p)$
N	= mode selection matrix, $n \times m$
n	= number of degrees of freedom
P	= random matrix, $2n \times n$
p	= number of experimental eigenvalues/eigenvectors, $(m = r = 2p)$
Q	= transformation matrix, $n \times n$
q	= weight factor
r	= number of sensor outputs, $(m = r = 2p)$
S_1	= partition of T^{-1}, $m \times 2n$
S_2	= partition of T^{-1}, $(2n - m) \times 2n$
s	= number of elements in experimental eigenvector
T	= transformation matrix, $2n \times 2n$
U	= analytical modal matrix, $n \times n$
$\boldsymbol{u}$	= control vector, $m \times 1$
$\boldsymbol{u}_i$	= measured components of $\boldsymbol{v}_i$, $s \times 1$
V	= matrix, $2n \times 2p$
$\tilde{\boldsymbol{v}}_i$	= ith measured eigenvector, $n \times 1 (s \times 1)$
$\boldsymbol{v}_i$	= ith partitioned eigenvector, $n \times 1$
v_{ia}	= ith achievable eigenvector, $n \times 1$
W	= achievable eigenvector matrix, $n \times p$
$\boldsymbol{w}$	= position vector, $n \times 1$
X	= solution of generalized algebraic Riccati equation, $m \times m$
y	= output vector, $r \times 1$
Z	= matrix, $m \times 2p$
α	= matrix defining A_i, $m \times 2p$
Λ	= diagonal eigenvalue matrix, $p \times p$
λ_i	= ith measured eigenvalue
σ	= matrix defining A_i, $2p \times m$
τ	= matrix defining A_i, $2p \times m$
$(\dot{\ })$	= differentiation with respect to time
$(\ )^*$	= complex conjugate transpose
$(\bar{\ })$	= complex conjugate
$\\| \ \\|_f$	= Frobenius norm

Introduction

HIGHLY accurate analytical models of flexible structures and machinery are required in order to predict dynamic performance. Due to the complexity of these structures, the finite-element method is typically used to develop the analytical model. The precision of the finite-element model may be improved to some acceptable level by increasing the number of degrees of freedom (DOF) included in the model. From a practical point of view, an acceptable level of precision is often defined to be when the eigenstructure of the system has converged in the lowest modes. However, the accuracy of the model in predicting the dynamics of the actual structure (when built) may be lacking due to uncertainties in material, damping, and joint properties, as well as fabrication-induced errors.

Confidence in the analytical model is often judged on how well the analytical modal properties (natural frequencies, damping ratios, and mode shapes) match the measured modal properties of the actual structure determined by experimental modal analysis. When the two sets of modal properties (or, alternatively, the eigenstructure of the system) are not in agreement, one may ask if it is possible to incorporate the measured modal data into the finite-element model to pro-

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*Assistant Professor, Department of Aerospace Engineering, Mechanics, and Engineering Science. Member AIAA.

†Graduate Research Assistant, Department of Mechanics.

duce an adjusted finite-element model with modal properties that closely match the experimental modal data.

Several different methodologies have been developed to determine the above-mentioned adjusted finite-element model. A comprehensive review of all developed methods would in itself constitute a lengthy survey paper. However, a brief review of previously developed work is necessary in order to view properly the new method developed in this paper. In general, the previously developed methods may be classified according to whether or not damping is considered in both the analytical model and experimental measurements. For the case where damping is neglected, several methods have been investigated that look solely at changing the stiffness matrix (analytical mass matrix assumed correct) to achieve the adjusted finite-element model.[1,2,3] Others have allowed both changes in the mass and stiffness matrices to achieve the adjusted finite-element model.[4,5] For cases where damping is accounted for, modification methods are based on orthogonality relations,[6] minimization of the error associated with measured modal parameters satisfying the analytical eigenvalue problem,[7] solution of the eigenvalue problem where the unmeasured modal properties (typically associated with higher modes of vibration) are assumed to be given by the original analytical model,[8] and through the use of eigenstructure assignment coupled with a nonlinear programming technique.[9]

In this paper, the use of a symmetric eigenstructure assignment technique is investigated to determine the adjusted finite-element model. It is assumed that an n-dimensional finite-element model of the structure exists. The damping may be nonproportional. The results of an experimental modal test of the structure are assumed available, and p eigenvalues and eigenvectors (possibly complex) have been identified, $p < n$. The identified eigenvectors are of length s, where $s < n$. Typically, both p and s are much smaller than n.

Problem Formulation

Structure/Control System Definition

Consider an n-DOF structural model with feedback control,

$$M\ddot{w} + D\dot{w} + Kw = B_0 u \qquad (1)$$

where M, D, and K are the $n \times n$ analytical mass, damping, and stiffness matrices, w is an $n \times 1$ vector of positions, B_0 is the $n \times m$ actuator influence matrix, u is the $m \times 1$ vector of control forces, and the overdots represent differentiation with respect to time. In addition, the $r \times 1$ output vector y of sensor measurements is given by

$$y = C_0 w + C_1 \dot{w} \qquad (2)$$

where C_0 and C_1 are the $r \times n$ output influence matrices. The control law is taken to be a general linear output feedback

$$u = Fy \qquad (3)$$

where F is the $m \times r$ feedback gain matrix. In Ref. 10, it is proven that if a system described by Eqs. (1) and (2) is controllable and observable, then by proper selection of F, max(m,r) closed-loop (controlled) eigenvalues can be assigned, max(m,r) closed-loop eigenvectors can be partially assigned with min(m,r) entries in each eigenvector being arbitrarily assigned.

The concept presented in Ref. 9 is to design a pseudocontroller such that the assigned closed-loop eigenvalues and eigenvectors match those determined experimentally. The controller is determined by specifying the matrices B_0, C_0, and C_1, and then calculating the feedback gain matrix F using eigenstructure assignment methods presented in Refs. 10 and 11 such that the measured modal parameters (or equivalently, eigenstructure) are assigned to the analytical model. Note that no control hardware is actually required, only the analytical technique of eigenstructure assignment is used. Substituting Eqs. (2) and (3) into (1), and moving the right-hand side to the left-hand side

$$M\ddot{w} + (D - B_0 F C_1)\dot{w} + (K - B_0 F C_0)w = 0 \qquad (4)$$

one can see that the matrix triple products $B_0 F C_0$ and $B_0 F C_1$ result in changes in the stiffness and damping matrices respectively. These triple products can then be viewed as perturbation matrices to the stiffness and damping matrices such that the adjusted finite-element model matches closely the experimentally measured modal properties. Unfortunately, these perturbation matrices are, in general, nonsymmetric when calculated using standard eigenstructure assignment techniques, thus yielding an adjusted stiffness and damping matrices that are also nonsymmetric.

In order to force the pseudocontroller to yield symmetric perturbation matrices, the approach taken in Ref. 9 was to pose an unconstrained optimization problem in which the $2nr$ ($r = m$ in Ref. 9) elements of C_0 and C_1 are the design variables, and the cost function is chosen to penalize the asymmetric portion of the perturbation matrices. If the cost function is driven to exactly zero, the resulting perturbation matrices are symmetric. However, for the case where the optimization problem results in a small, but nonzero value for the objective function, the asymmetric portion of the perturbations are ignored. This results in the analytical model not exactly matching the experimental data. However, the entire optimization procedure can then be repeated using the newly adjusted finite-element model to calculate new perturbation matrices. Therefore, the search for a symmetric pseudocontroller requires the solution of an unconstrained minimization problem embedded within an iterative scheme. As stated in Ref. 9, convergence of this iterative scheme is not assured. A limitation to the practical application of this method is that the number of design variables is linearly dependent on n, the size of the finite-element model. For structures where n is relatively large such that the finite-element model has high precision, the number of design variables ($2nr$) may become too large for current unconstrained optimization algorithms to handle.

Symmetric Eigenstructure Assignment

In the new symmetric eigenstructure assignment method proposed in this paper, a noniterative technique is developed to determine the symmetric perturbation matrices. The computational requirements of the method are reduced in comparison to the iterative unconstrained optimization procedure employed in Ref. 9. Additionally, the new method provides insight into the numerical method of Ref. 9.

The "symmetric" eigenstructure assignment method can be divided into two separate steps. Due to practical testing limitations, the dimension of the experimentally determined eigenvectors is usually substantially less than that of the analytical eigenvectors. One solution to this problem is to employ a model reduction technique such that the reduced dimension of the analytical model matches that of the experimentally measured eigenvector. The alternate approach, which is employed in this work, is to expand the measured eigenvector to the size of the analytical eigenvector. Using this approach, it is not always possible to assign exactly to the adjusted finite-element model the eigenvectors that were measured. Therefore, in step one, the best "achievable" eigenvectors, in a least-squares sense, are determined. As shown in Ref. 11, the achievable $n \times 1$ eigenvectors v_{ia} of the system defined by Eq. (1) must lie in the subspace defined by

$$L_i = (M\lambda_i^2 + D\lambda_i + K)^{-1} B_0 \quad i = 1,\ldots,p \qquad (5)$$

where λ_i is the measured eigenvalue associated with the ith measured eigenvector $\tilde{v}_i$. The vector $\tilde{v}_i$ is an $n \times 1$ representation of the s components of the measured eigenvector. The s components are placed in $\tilde{v}_i$ according to the finite-element node number corresponding to the measured component location. All other elements of $\tilde{v}_i$ (which represent unmeasured components) are set to zero. Define a transformation Q that reorders the eigenvector such that those positions of the analytical model that correspond to measured eigenvector locations now comprise the top portion of the vector

$$v_i = Q\tilde{v}_i = \underset{1}{\begin{bmatrix} u_i \\ d_i \end{bmatrix}} \begin{matrix} s \\ n-s \end{matrix} \tag{6}$$

The u_i would then be the measured $s \times 1$ eigenvector, and d_i is the $(n-s) \times 1$ vector of free entries. Applying the same transformation to Eq. (5), one can partition the subspace as

$$QL_i = \underset{m}{\begin{bmatrix} \tilde{L}_i \\ D_i \end{bmatrix}} \begin{matrix} s \\ n-s \end{matrix} \tag{7}$$

The best achievable eigenvector, v_{ia}, in a least-squares sense, is then given by

$$v_{ia} = L_i[\tilde{L}_i^* \tilde{L}_i]^{-1} \tilde{L}_i^* u_i \quad i = 1,...,p \tag{8}$$

For this vector to exist, the dimension of the measured eigenvector must be greater than or equal to the number of pseudoactuators ($s \geq m$). If this condition is not met ($s < m$), then it is always possible to assign the measured eigenvectors exactly. It is interesting to note that the equations defining the best achievable eigenvectors developed in eigenstructure assignment theory[11] are exactly the relationships developed in Ref. 5 relating measured and unmeasured components of the eigenvector. Summarizing, after step one is completed, there are p sets of experimental eigenvalues and experimental achievable eigenvectors, which the adjusted finite-element model should match.

In step two, the feedback gain matrix, F, is calculated such that the adjusted finite-element model matches the experimental modal data. The matrix F is given as[11]

$$F = [Z - A_U V]\left\{[C_0 \mid C_1]\begin{bmatrix} W & \bar{W} \\ W\Lambda & \bar{W}\bar{\Lambda} \end{bmatrix}\right\}^{-1} \tag{9}$$

where

$$A = \begin{bmatrix} \underset{n}{0} & \underset{n}{I} \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{matrix} n \\ n \end{matrix} \tag{10a}$$

$$B = \underset{m}{\begin{bmatrix} 0 \\ M^{-1}B_0 \end{bmatrix}} \begin{matrix} n \\ n \end{matrix} \tag{10b}$$

$$T = [\underset{m}{B} \mid \underset{2n-m}{P}]^{2n} \tag{10c}$$

$$\tilde{A} = T^{-1}AT = \underset{2n}{\begin{bmatrix} A_u \\ A_l \end{bmatrix}} \begin{matrix} m \\ 2n-m \end{matrix} \tag{10d}$$

$$\tilde{B} = T^{-1}B = \underset{m}{\begin{bmatrix} I_m \\ 0 \end{bmatrix}} \begin{matrix} n \\ 2n-m \end{matrix} \tag{10e}$$

$$V = T^{-1}\begin{bmatrix} W & \bar{W} \\ W\Lambda & \bar{W}\bar{\Lambda} \end{bmatrix} \tag{10f}$$

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_p) \tag{10g}$$

$$W = \underset{p}{[v_{1a} \mid v_{2a} \mid \cdots \mid v_{pa}]}\, n \tag{10h}$$

$$Z = S_1\left\{\begin{bmatrix} W \\ W\Lambda \end{bmatrix}\Lambda \,\middle|\, \begin{bmatrix} \bar{W} \\ \bar{W}\bar{\Lambda} \end{bmatrix}\bar{\Lambda}\right\} \tag{10i}$$

$$T^{-1} = \underset{2n}{\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}} \begin{matrix} m \\ 2n-m \end{matrix} \tag{10j}$$

where the partition matrix P in matrix T is arbitrary as long as T^{-1} exists. Therefore, P must be of full rank (rank $= 2n - m$), and the columns must be linearly independent of B. It is possible to select P such that the inversion of T can be accomplished by the inversion of matrices of dimension $n/2$.

The adjusted stiffness, K_a, and damping, D_a, matrices are then given by

$$K_a = K - B_0 F C_0 \qquad D_a = D - B_0 F C_1 \tag{11}$$

At this point, B_0, C_0, and C_1 are arbitrary, and a general selection would result in nonsymmetric perturbation matrices. For the perturbations to be symmetric, the following conditions must be met

$$B_0 F C_i = C_i^T F^T B_0^T \quad i = 0,1 \tag{12}$$

The following two assumptions are made. First, the number of pseudosensors and pseudoactuators are taken to be equal to twice the number of measured modes ($m = r = 2p$). This is a requirement such that the inverse of certain matrices exist. This assumption implies that only a maximum of $n/2$ measured eigenvalues and eigenvectors can be placed. This does not normally present a practical difficulty. Secondly, the C_i are written as

$$C_i = G_i B_0^T \quad i = 0,1 \tag{13}$$

where G_i are $m \times m$ matrices and G_i^{-1} exists. Of course, it is recognized that not all possible C_i can be obtained from Eq. (13) once a B_0 has been selected. Substituting Eq. (13) into Eq. (12), the symmetry statement can be written as

$$FG_i = G_i^T F^T = G_i^* F^* \tag{14}$$

Substituting Eq. (9) into Eq. (14) and expanding, one can obtain a necessary but not sufficient condition on the G_i for symmetric perturbation matrices in terms of a generalized algebraic Riccati equation

$$A_1 X + X A_2 + X A_3 X + A_4 = 0 \tag{15}$$

where

$$X = G_1^{-1} G_0 \tag{16a}$$

$$A_1 = [\sigma^* \alpha^{-1} \sigma^{-1}(\alpha^* \tau^* - \tau\alpha) - \tau^*]\sigma^{-*} \tag{16b}$$

$$A_2 = \tau^* \alpha^{-1} \sigma^{-1} \alpha^* \tag{16c}$$

$$A_3 = \tau^* \alpha^{-1} \sigma^{-1}(\alpha^* \tau^* - \tau\alpha)\sigma^{-*} \tag{16d}$$

$$A_4 = \sigma^* \alpha^{-1} \sigma^{-1} \alpha^* - I_m \tag{16e}$$

$$\tau = \begin{bmatrix} W^* B_0 \\ \bar{W}^* B_0 \end{bmatrix} \qquad \sigma = \begin{bmatrix} \Lambda W^* B_0 \\ \bar{\Lambda} W^* B_0 \end{bmatrix} \qquad \alpha = Z - A_l V \tag{16f}$$

It should be noted that although the matrices τ, σ, and α are complex, the $A_i, i = 1,4$, are all real. Equation (15) is not the standard algebraic Riccati equation because $A_1 \neq A_2^T$.

However, solution techniques developed for the standard Riccati equation[12,13,14] can be used to solve Eq. (15). In particular, the method used is based on the developments of Refs. 13 and 14.

The solution of Eq. (15) requires the calculation of the eigenvectors of a $2m \times 2m$ Euler matrix denoted as E

$$E = \begin{bmatrix} A_2 & A_3 \\ -A_4 & -A_1 \end{bmatrix} \tag{17}$$

The eigenvectors of E are denoted by $\boldsymbol{a}_i$, $i = 1{,}2m$ and further partitioned as

$$\boldsymbol{a}_i = \begin{bmatrix} \boldsymbol{b}_i \\ \boldsymbol{c}_i \end{bmatrix} \begin{matrix} m \\ m \end{matrix} \quad (1) \tag{18}$$

The solutions of Eq. (15) can then be written for all possible permutations of $\boldsymbol{a}_i$ as[14]

$$X = [\boldsymbol{c}_1 \cdots \boldsymbol{c}_m][\boldsymbol{b}_1 \cdots \boldsymbol{b}_m]^{-1} \tag{19}$$

where the inverse is assumed to exist for certain combinations of eigenvectors.

Updating Stiffness and Damping Matrices

In general, multiple real and complex solutions of Eq. (15) exist. However, because the G_i are required to be real to yield a physically meaningful solution, only the real solutions of Eq. (15) are sought. All possible real solutions are obtained for X by making restrictions on the eigenvectors, $\boldsymbol{a}_i$, used in Eq. (19). Necessary and sufficient conditions for the solution X to be real are a) all eigenvectors $\boldsymbol{a}_1,...,\boldsymbol{a}_m$ used in the construction of X are real, or b) if $\boldsymbol{a}_i$ corresponding to the eigenvalue λ_i, with $Im(\lambda_i) \neq 0$, is used to construct the solution X, then $\bar{\boldsymbol{a}}_i$ corresponding to $\bar{\lambda}_i$ must also be included in the solution. It can be shown that there are typically, at most, $m!/[(m/2)!]$ real solutions to Eq. (15).[14] Note that the size of the required calculations is only dependent on the number of measured sets of modal properties, not the size of the analytical finite-element model.

With all possible real solutions of X in hand, it is now required to determine G_0 and G_1. It can be shown that the product FG_0 and FG_1 is not explicitly dependent on the choice of G_0 or G_1, but is only dependent on X.[15] Therefore, either G_0 (or G_1) can be chosen arbitrarily, as long as its inverse exists. Then, G_1 (or G_0) is calculated from the relationship $X = G_1^{-1}G_0$. To reduce the computational requirements, G_1 is chosen to be the $m \times m$ identity matrix. Then, using Eqs. (9), (10), and (13), the feedback gain matrix F is calculated. The adjusted stiffness and damping matrices are then calculated from Eq. (11).

Thus, for each real solution X, we obtain different adjusted stiffness and damping matrices. A rationale for choosing the "best" adjusted stiffness and damping matrices is now developed. Because Eq. (15) is only a necessary, but not sufficient condition for symmetry, some of the adjusted matrices are not symmetric and, therefore, are immediately eliminated as candidate solutions. Additionally, some of the adjusted stiffness and damping matrices no longer have the same definiteness as the original stiffness and damping matrices (indicating relatively large perturbation matrices), and thus are also eliminated as candidate solutions. If no matrix set remains, then a different B_0 must be selected. The final selection for the adjusted stiffness and damping matrices is then made by choosing from the remaining matrix sets the set that results in the minimum value of the evaluation function

$$J = q\|K - K_a\|_f + \|D - D_a\|_f \tag{20a}$$

$$q = (\|D\|_f)/(\|K\|_f) \tag{20b}$$

This evaluation function selects the adjusted stiffness and damping matrix set that results in a minimum change of the analytical model. Note that this is not an optimization problem; the function J is simply a way to evaluate the best solution from the total solution set. Because of the different scales in the numerical entries of the damping and stiffness matrices, a scale factor q is used such that changes in the damping and stiffness matrices are given equal weight. The existence of multiple symmetric solutions to the problem raises the question to which solution the numerical method presented in Ref. 9 would converge. Because the method uses a gradient-based minimization algorithm, it is quite possible that the final solution may be one that changes the definiteness of either the damping or stiffness matrix.

Selection of B_0

At this point, the control influence matrix B_0 is a free variable. The only condition placed on B_0 is that it must be of full rank. Therefore, different selections for B_0 may possibly result in different adjusted damping and stiffness matrices. Thus, a rationale for selecting B_0 is required.

Modal Matrix Method

The modal matrix selection method is based on the premise that it is desirable to affect only the eigensolution of the analytical model corresponding to those eigenvalues/vectors that have been experimentally measured. Therefore, B_0 should be selected such that the unmeasured modes of the analytical model are uncontrollable. Of course, because the original analytical model may have nonproportional damping, the system modes may be coupled to some extent. However, with the mass normalized modal matrix $U = [\boldsymbol{u}_1 \boldsymbol{u}_2 ... \boldsymbol{u}_n]$, where the $\boldsymbol{u}_i$ are the eigenvectors of the original undamped analytical model, the selection of B_0 is chosen as

$$B_0 = [U^T]^{-1}N = MUN \tag{21}$$

where N is a $n \times m$ matrix. Substituting B_0 as defined by Eq. (21) into Eq. (1) and transforming to modal space, it can be seen that N is the modal control influence matrix. The elements of N are thus selected depending on which modes of the analytical model are to be adjusted. If the jth mode of the analytical model is to be adjusted, the jth row of N should contain nonzero elements. Conversely, if the jth mode of the analytical model is not to be adjusted, the jth row of N should contain zeros. The calculation of B_0 as defined by Eq. (21) requires the calculation of the full analytical modal matrix. However, because the objective in selecting B_0 is to have the control influence only the analytical eigenstructure corresponding to that which is measured, it is sufficient to replace the $\boldsymbol{u}_i$ in Eq. (21) corresponding to the unmeasured eigenstructure with zero vectors. Therefore, only those analytical eigenvectors that correspond to the measured eigenvectors must be determined. Thus, only p eigenvectors of the original analytical model must be determined. The N and modified U matrices as defined above will result in B_0 having rank $m/2$, whereas B_0 is required to have full rank m. Therefore, it is required that a few elements in $m/2$ zero columns of U and zero rows of N be replaced by random and arbitrarily small (epsilon) numbers such that B_0 is of full rank. Therefore, some of the unmeasured eigenstructure of the analytical model will be slightly affected by the control, although in practice the effect is not noticeable.

Nonlinear Optimization

The selection for B_0 can also be driven by posing a nonlinear optimization problem with the nm terms of B_0 acting as design variables. Of course, this greatly increases the computational burden of the proposed algorithm. However, it should be noted that the algorithm developed in Ref. 9 requires optimization of $2nr$ $(r = m)$ design variables. The objective function of the optimization problem is to minimize a weighted difference between the original and adjusted stiffness and damping matrices

$$\min_{\substack{\text{wrt}\\ B_0}} J = q\|K - K_a\|_f + \|D - D_a\|_f \tag{22a}$$

$$q = (\|D\|_f)/(\|K\|_f) \tag{22b}$$

A good initial starting guess for the optimization can be obtained using the modal matrix method. The number of design variables nm can be reduced to a practical number by using a judiciously chosen subset of the nm possible variables as actual design variables. This choice can be guided by using the modal matrix selection method. As an example, the nonzero elements of N (not including the epsilon elements) could be chosen as design variables. However, the additional computations associated with the optimization procedure may outweigh any practical benefits in practice.

Examples

Example #1

Consider a five-DOF system modeled analytically with mass, damping, and stiffness matrices given by

$$M = \text{diag}(1,2,5,4,3)$$

$$D = \begin{bmatrix} 11 & -2 & 0 & 0 & 0 \\ & 14 & -3.5 & 0 & 0 \\ & & 13.0 & -1.2 & 0 \\ & \text{SYM} & & 13.5 & -4 \\ & & & & 15.4 \end{bmatrix}$$

$$K = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ & 120 & -35 & 0 & 0 \\ & & 80 & -12 & 0 \\ & \text{SYM} & & 95 & -40 \\ & & & & 124 \end{bmatrix}$$

The system given by M, D, and K represents the original, proportionally damped finite-element model of the structure. The model used to simulate the consistent experimental data is given by $M_e = M$, $D_e = D$ and stiffness K_e given by

$$K_e = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ & 120 & -35 & 0 & 0 \\ & & 70 & -12 & 0 \\ & \text{SYM} & & 95 & -40 \\ & & & & 124 \end{bmatrix}$$

Note the difference between K and K_e is in the (3,3) element. The eigensolution of the experimental model is used to create the experimental modal data. It is assumed that only the fundamental mode characteristics are experimentally determined and that only the second and third components of the eigenvector are measured

$$\lambda_1 = -1.116 + 3.057i$$

$$v_1 = \lfloor x \quad 0.3708 + 0.0048i \quad 1 \quad x \quad x \rfloor^T$$

where the x represents the unmeasured components of the eigenvector. As is the case with most experimental data, the measured eigenvector is complex, indicating that the experimental structure exhibits nonproportional damping. For comparison, the fundamental mode characteristics of the original finite-element model are given by

$$\lambda_1 = -1.113 + 3.320i$$

$$\boldsymbol{v}_1 = [0.0878 \quad 0.3850 \quad 1.0 \quad 0.4347 \quad 0.1994]^T$$

There is an approximately 9% error between the measured and analytical fundamental frequency.

To match the measured modal parameters, the number of required pseudoactuators is two ($m = r = 2p = 2$). The control influence matrix B_0 was chosen using the modal matrix method such that only the fundamental mode is controlled

$$B_0 = \begin{bmatrix} 0.0828 & 0.0353 \\ 0.2878 & 0.3097 \\ 2.0158 & 2.0114 \\ 0.6993 & 0.6995 \\ 0.2406 & 0.2406 \end{bmatrix}$$

The Euler block matrices A_i, $i = 1,4$, are calculated to be

$$A_1 = \begin{bmatrix} -917.4 & 909.6 \\ -916.3 & 908.5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 969.4 & -935.1 \\ 1014.6 & -978.7 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.2220 & 0.1957 \\ -0.2563 & 0.2288 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 3.2974 & -3.1806 \\ 3.2927 & -3.1760 \end{bmatrix} \times 10^5$$

The solution of the generalized Riccati equation, X, is then given as

$$X = \begin{bmatrix} -7131.2 & -7870.7 \\ 8304.9 & -8961.3 \end{bmatrix}$$

The feedback gain matrix F is then calculated to be

$$F = \begin{bmatrix} -0.3259 & 0.2956 \\ 0.2956 & -0.2640 \end{bmatrix}$$

The resulting adjusted stiffness and damping matrices, K_a and D_a, are given as

$$D_a = \begin{bmatrix} 11.00 & -2.00 & 0.00 & 0.00 & 0.00 \\ & 14.00 & -3.50 & 0.00 & 0.00 \\ & & 13.00 & -1.20 & 0.00 \\ \text{SYM} & & & 13.50 & -4.00 \\ & & & & 15.40 \end{bmatrix}$$

$$K_a = \begin{bmatrix} 99.55 & -20.5 & -4.64 & -1.60 & -0.55 \\ & 120.4 & -34.0 & 0.37 & 0.13 \\ & & 72.63 & -14.4 & -0.83 \\ \text{SYM} & & & 94.22 & -40.3 \\ & & & & 123.9 \end{bmatrix}$$

Inspection shows that the damping matrix has not been modified, i.e., the original damping matrix is correct. Although all the elements of the stiffness matrix have been adjusted, the algorithm has concentrated the major change in the proper location.

Next, using the B_0 above as an initial guess, the unconstrained optimization problem for selecting B_0 described by Eq. (22) was then solved. This resulted in a solution

$$B_0 = \begin{bmatrix} 0.0397 & 0.0429 \\ 0.3561 & 0.3635 \\ 3.2744 & -0.8633 \\ 0.4810 & 0.4881 \\ 0.2127 & 0.2170 \end{bmatrix}$$

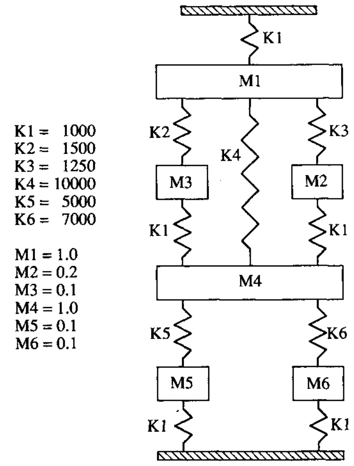

Fig. 1 Six-DOF model taken from Kammer.[2]

with adjusted matrices D_a as before and K_a given as

$$K_a = \begin{bmatrix} 100.0 & -20.0 & 0.01 & 0.00 & 0.00 \\ & 120.0 & -35.0 & 0.00 & 0.00 \\ & & 70.01 & -12.0 & -0.00 \\ \text{SYM} & & & 95.00 & -40.0 \\ & & & & 124.0 \end{bmatrix}$$

Therefore, combined with the unconstrained optimization problem, the algorithm was able to almost exactly reproduce the correct stiffness matrix.

Example #2

Consider the test problem investigated by Kabe[16] and Kammer,[2] as illustrated in Fig. 1. The eigensolution of the six-DOF model using the exact coefficients was used to simulate the experimental data. Stiffness terms were then corrupted to simulate the inaccurate original analytical model. The upper triangular exact and corrupted stiffness matrix coefficients are listed in Table 1. Note that some of the stiffness terms have been corrupted by as much as 30%. Because the symmetric eigenstructure assignment (SEA) method requires the system to be damped, a small amount of artificial damping was added (less than 0.2% modal damping for the fundamental mode) for the SEA calculations.

The adjusted stiffness coefficients found by the Baruch, Kabe, Projector Matrix (PM), and SEA methods using only the experimental fundamental frequency and mode shape are given in Table 1. The results for the Baruch, Kabe, and PM methods were taken directly from the paper by Kammer.[2] For the SEA method, the B_0 matrix was chosen using the modal matrix method. Inspection of the table shows that all four methods obtain similar results. Note that the Baruch and SEA methods introduce load paths that are not in the original model. However, the magnitude of these load-path coefficients is relatively small. It should be noted that the computational burden of the Kabe and PM methods is much greater than that required by the Baruch and SEA methods. In fact, the computational burden of the Kabe and PM methods makes the application of these methods impractical for large-scale finite-element models.

Summary

In conclusion, a new method has been developed to incorporate measured experimental modal data into an analytical finite-element model with nonproportional damping, such that the adjusted finite-element model more closely matches the experimental results. The method is based on the development of a new symmetric eigenstructure assignment method.

The proposed algorithm requires the solution of a generalized algebraic Riccati equation, whose size is dependent only on the number of experimentally measured modes. An algorithm based on incorporating the original algorithm within an unconstrained optimization problem has been shown to produce improved results. However, the resulting increase in computational burden may outweigh the improved results in actual practice.

Table 1 Corrupted, exact, and adjusted stiffness coefficients

Coeff. location	Corrupt coeff.	Adjusted stiffness coefficient				Exact coeff.
		Baruch	Kabe	PM	SEA	
1,1	15750	15576	15543	15540	15748	13750
1,2	−1300	−1314	−1299	−1299	−1301	−1250
1,3	−1300	−1303	−1298	−1298	−1300	−1500
1,4	−12000	−11939	−12001	−11999	−12011	−10000
1,5	0	−22	0	0	2	0
1,6	0	−63	0	0	4	0
2,2	2150	2152	2160	2159	2150	2250
2,3	0	2	0	0	0	0
2,4	−850	−813	−849	−848	−840	−1000
2,5	0	−3	0	0	2	0
2,6	0	−12	0	0	5	0
3,3	2150	2151	2163	2161	2150	2500
3,4	−850	−829	−849	−848	−848	−1000
3,5	0	−1	0	0	1	0
3,6	0	−5	0	0	1	0
4,4	22900	23184	23310	23326	23259	24000
4,5	−4200	−4202	−4208	−4211	−4274	−5000
4,6	−5000	−5041	−5307	−5056	−5179	−7000
5,5	5100	5098	5066	5071	5115	6000
5,6	0	−6	0	0	36	0
6,6	5900	5890	5787	5809	5989	8000

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